# Single Particle Excitations in the Lattice $E_8$ Ising Model

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### Abstract

We present analytic expressions for the single particle excitation energies of the 8 quasi-particles in the lattice  $E_8$  Ising model and demonstrate that all excitations have an extended Brillouin zone which, depending on the excitation, ranges from  $0 < P < 4\pi$  to  $0 < P < 12\pi$ . These are compared with exact diagonalizations for systems through size 10 and with the  $E_8$  fermionic representations of the characters of the critical system in order to study the counting statistics.

# 1. Introduction

Several years ago Bazhanov, Nienhuis and Warnaar [1] demonstrated that the the solution of the dilute  $A_3$  model of Warnaar, Nienhuis and Seaton [2]-[3] may be reduced to the Bethe's Ansatz equation based on the group  $E_8$  given in [4]-[6]. This thus provides a lattice realization of the  $E_8$  continuum field theory found by Zamolodchikov [7]-[8] in 1989 to be in the same universality class as the Ising model in a magnetic field at  $T = T_c$ .

The reduction in [1] to the  $E_8$  equations is based on a conjecture for the allowed string type solutions of the Bethe's Ansatz equations for the dilute  $A_3$  model and this conjecture was motivated by a numerical study of the system at criticality in the sector l=1 which contains the ground state. This study gives a set of counting rules for the spectrum which exactly reproduces the  $E_8$  fermionic representation of the character  $\chi_{1,1}^{(3,4)}(q)$  of the Ising model which was conjectured in [9]-[10] and proven in its polynomial generalization in [11].

However, there is as yet no satisfactory conjecture for the string content at criticality of all states in the sectors l=2 and l=3 (which correspond to the characters  $\chi_{1,2}^{(3,4)}(q)$  and  $\chi_{1,3}^{(3,4)}(q)$ ) and a recent study [12]-[13] of numerical solutions to the non-critical Bethe's equations at zero momentum indicates that the string identification may at times change as the system moves away from criticality. We make a two fold study of these questions here by numerically computing the single particle excitations for finite chains of sizes up through 10 and by comparing these results with the single particle dispersion relations which are obtained by extending the computations of [1]. We find that all eight quasiparticles exhibit the phenomenon of an extended Brillouin zone such as was first explicitly seen in the anti-ferromagnetic three state Potts spin chain [14] and that there are no genuine one particle states in the two sectors l=2 and 3.

We present the dispersion relations in Sec. 2, the numerical study in Sec. 3 and discuss our findings in Sec. 4. We follow the notation of ref. [1] and refer the reader to that paper for the explicit Boltzmann weights which define the model. We will here be concerned only with the Hamiltonian version of the model.

## 2. Single Particle Dispersion Relations and Characters

In [1] and [4] the single particle excitations were studied in the limit  $L \to \infty$  and in particular the Fourier transform of the single particle excitations in sector l=1 is computed. We have inverted the transform to find the results given below. The computations involve some tedious algebra and moreover for the non-critical case the required integrals do not seem to be in the literature. The details will be given elsewhere.

For the critical case we have as a function of rapidity u

$$P_{j}(u) = \sum_{a} \left[ \pi + 2 \arctan\left(\frac{\sinh u}{\sin(a\pi/30)}\right) \right]$$

$$e_{j}(u) = A \frac{dP_{j}(u)}{du} = A \cosh u \sum_{a} \frac{\sin(a\pi/30)}{\cosh^{2} u - \cos^{2}(a\pi/30)}$$

$$(2.1)$$

where A = 32/15,  $-\infty < u < \infty$ , the number thirty has the significance of being the Coxeter number of  $E_8$  and a takes on the following values as a function of j (we use the labeling of the Cartan matrix  $E_8$  shown in Fig. 1 and indicate the identification of particles of [1] in parentheses)

The numbers a in this table agree with the corresponding numbers for the scattering of a particle of type 1 with a particle of one of the 8 types of the  $E_8$  field theory as given in [15] and have group theoretical and geometrical interpretations [16]-[18]. They also follow from (4.19) of [1]. The numbers  $\Delta P$  are  $2\pi$  times the elements in the first column of the inverse Cartan matrix.

For the non-critical case in which the nome  $q_B$  of the Boltzmann weights of [1] (which is essentially the magnetic field) is not zero we have the following generalization which reduces to (2.1) when  $q \to 0$  (and the modulus  $k \to 0$ )

$$P_{j}(u,q) = \sum_{a} \left[ \pi + 2 \arctan\left(\frac{i \operatorname{sn} iu}{\operatorname{sn}(aK/15)}\right) \right]$$

$$e_{j}(u,q) = A(q) \frac{dP_{j}(u)}{du} = A(q) \operatorname{dn}iu \operatorname{cn} iu \sum_{a} \frac{\operatorname{sn}(aK/15)}{\operatorname{cn}^{2}iu - \operatorname{cn}^{2}(aK/15)}$$
(2.3)

where  $q = q_B^{16/15}$ ,  $A(q) = 2AK(q)/\pi$ , K(q) is the complete elliptic integral of the first kind and -K' < u < K'. This expression is in fact a universal form for all the Bethe's Ansatz models based on a simply laced Lie algebra, the only difference being that the values of a, here given by (2.2), vary from model to model.

The minimum in the energy occurs  $u=\pm K'$  where P=0 (or the value  $\Delta P$  given in (2.2)). Thus we find

$$e_j(P=0,q) = kA(q)\sum_a \text{sn}(aK/15)$$
 (2.4)

and in particular

$$\lim_{k \to 0} e_j(P = 0, q)/kA(q) = \sum_{a} \sin(a\pi/30). \tag{2.5}$$

It may be verified that these values coincide with the components of the Perron-Frobenius eigenvector of the  $E_8$  Cartan matrix (as given, for example, in [19].)

When  $q \to 1 \ (k \to 1)$  we find that

$$e_j(u,q) \to A(q) \sum_a 1$$
 (2.6)

where the normalizing constant is diverging. Thus using (2.2) we see that the single particle states for particles 1 and 2 become degenerate as  $q \to 1$  (and similarly for particles 3 and 4). For the remaining particles degeneracies with multi-particle states also occur.

In principle u can be eliminated between the two expressions in (2.3) to produce a polynomial relation between  $e_j$  and  $P_j$ . However here we have done the elimination numerically and present the results in Fig. 2 for  $q_B = 0$  and in Fig. 3 for  $q_B = 0.2$ . In Fig. 3 we

see that the labeling we have used corresponds to the ordering of eigenvalues at P=0 which is the same for all q>0. It is to be explicitly remarked that the dispersion relations do not have the restriction  $0 < P < 2\pi$  but in all cases have the larger momentum range  $0 < P < \Delta P$  where  $\Delta P$  is given in (2.2). This is the phenomenon of the extended Brillouin zone scheme. This is a very general phenomenon in integrable models first explicitly seen in [14] for the three state anti-ferromagnetic Potts spin chain.

We also wish to make contact with the characters of conformal field theory and finite size computations. The fermionic representation of characters of the Ising model in the  $E_8$  basis is given [9]-[10] in terms of the fermionic form

$$\sum_{n_1,\dots,n_8=0}^{\infty} \frac{q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n} - \mathbf{A} \cdot \mathbf{n}}}{(q)_{n_1}\dots(q)_{n_8}},$$
(2.7)

where  $(q)_n = \prod_{j=1}^n (1-q^j)$ ,  $C_{E_8}$  is the Cartan matrix of the Lie algebra  $E_8$  given by the incidence matrix of Fig. 1 (where we use the labeling of [9]) and we trust that q will not be confused with the nome of the elliptic functions. Explicitly

$$C_{E_8}^{-1} = \begin{pmatrix} 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 4 & 4 & 5 & 6 & 7 & 8 & 10 \\ 3 & 4 & 6 & 6 & 8 & 8 & 10 & 12 \\ 3 & 5 & 6 & 8 & 9 & 10 & 12 & 15 \\ 4 & 6 & 8 & 9 & 12 & 12 & 15 & 18 \\ 4 & 7 & 8 & 10 & 12 & 14 & 16 & 20 \\ 5 & 8 & 10 & 12 & 15 & 16 & 20 & 24 \\ 6 & 10 & 12 & 15 & 18 & 20 & 24 & 30 \end{pmatrix}$$

$$(2.8)$$

For the characters  $\chi_{1,s}^{(3,4)}(q)$  (normalized to  $\chi_{1,s}^{(3,4)}(0)=1$ ) it was conjectured in [9] and proven in [11] that

$$\chi_{1,1}^{(3,4)}(q) = \sum_{n_1,\dots,n_8=0}^{\infty} \frac{q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n}}}{(q)_{n_1}\dots(q)_{n_8}}$$
(2.9)

and in [10] it was conjectured on the basis of computer studies that

$$\chi_{1,1}^{(3,4)}(q) + \chi_{1,2}^{(3,4)}(q) = \sum_{n_1,\dots,n_8=0}^{\infty} \frac{q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n} - \mathbf{A}^{(1)} \cdot \mathbf{n}}}{(q)_{n_1} \dots (q)_{n_8}} \text{ with } A_j^{(1)} = (C_{E_8}^{-1})_{1,j}$$

$$\chi_{1,1}^{(3,4)}(q) + \chi_{1,2}^{(3,4)}(q) + \chi_{1,3}^{(3,4)}(q) = \sum_{n_1,\dots,n_8=0}^{\infty} \frac{q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n} - \mathbf{A}^{(2)} \cdot \mathbf{n}}}{(q)_{n_1} \dots (q)_{n_8}} \text{ with } A_j^{(2)} = (C_{E_8}^{-1})_{2,j}.$$

$$(2.10)$$

We also note that the character (2.9) is derived from the lattice quasiparticle spectrum

$$E - E_{GS} = \sum_{j=1}^{8} \sum_{i=1}^{n_j} e_j(P_j^i)$$
 (2.11)

$$P = \sum_{j=1}^{8} \sum_{i=1}^{n_j} P_j^i \pmod{2\pi}, \tag{2.12}$$

where  $e_j(P)$  are single particle dispersion relations, we have the fermionic restriction

$$P_j^i \neq P_j^k$$
 for  $i \neq k$  and all  $j$  (2.13)

and the momenta are chosen from the set

$$P_j^i \in \left\{ P_j^{\min}(\mathbf{n}), \ P_j^{\min}(\mathbf{n}) + \frac{2\pi}{L}, \ P_j^{\min}(\mathbf{n}) + \frac{4\pi}{L}, \ \dots, \ P_j^{\max}(\mathbf{n}) \right\}, \tag{2.14}$$

with

$$P_j^{\min}(\mathbf{n}) = \frac{2\pi}{L} \left[ (\mathbf{n} C_{E_8}^{-1})_j + \frac{1}{2} (1 - n_j) \right]$$
 (2.15)

and

$$P_j^{\text{max}}(\mathbf{n}) = -P_j^{\text{min}}(\mathbf{n}) + 2\pi (C_{E_8}^{-1})_{1,j} .$$
 (2.16)

## 3. Finite Size Study

We have numerically determined the Hamiltonian eigenvalues for chains up through size 10. We have determined the sector l from the property that at q=0 eigenvectors in the sector l=2(l=1,3) are antisymmetric (symmetric) under the interchange  $1\leftrightarrow 3$  of the states in the basis of [2], and by a similar symmetry property under the interchange  $2\leftrightarrow (1+3)/\sqrt{2}$  that distinguishes l=1 from l=3. We present these data in Figs. 4–7 where we also compare with the  $L\to\infty$  formula (2.3).

In Fig. 4 we plot at q=0 the eigenvalues which behave as 2A(q) as  $q \to 1$ . These include all the single particle states for particles 1 and 2 in the sector l=1 which are allowed by (2.14). These states are indicated with triangles. The remaining states are to be found in the sectors l=2 and l=3. We note that while the number of states in l=1 grows proportionally with L that there are only 4 states with l=2 and 2 states

with l=3. The two l=2 states of particle 1 at  $P=0,2\pi/10$  are accounted for by subtracting (2.9) from the first equation of (2.10) to write

$$\chi_{1,2}^{(3,4)}(q) = \sum_{n_1,\dots,n_8=0}^{\infty} \frac{q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n} - \mathbf{A}^{(1)} \cdot \mathbf{n}} - q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n}}}{(q)_{n_1}\dots(q)_{n_8}}$$
(3.1)

and by considering the  $n_1 = 1, n_2, ..., n_8 = 0$  term of the sum to find  $(1-q^2)/(1-q) = 1+q$ . Similarly, the two l = 2 states of particle 2 at  $P = 4\pi/10, 6\pi/10$  are obtained from (3.1) by setting  $n_1 = 0, n_2 = 1, n_3, ..., n_8 = 0$  to obtain  $(q^2 - q^4)/(1 - q) = q^2 + q^3$ . and the two l = 3 states of particle 2 at  $P = 0, 2\pi/10$  are obtained by first using (2.10) to write

$$\chi_{1,3}^{(3,4)}(q) = \sum_{n_1,\dots,n_8=0}^{\infty} \frac{q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n} - \mathbf{A}^{(2)} \cdot \mathbf{n}} - q^{\mathbf{n}C_{E_8}^{-1}\mathbf{n} - \mathbf{A}^{(1)} \cdot \mathbf{n}}}{(q)_{n_1}\dots(q)_{n_8}}$$
(3.2)

and then setting  $n_1 = 0, n_2 = 1, n_3, \dots, n_8 = 0$  to obtain  $(1 - q^2)/(1 - q) = 1 + q$ .

In Fig. 5 we do a similar study for the states which behave as 4A(q) as  $q \to 1$ . The states which are the single particle excitations for particles 3 and 4 in l=1 are again marked with triangles and of the remaining states there are 9 in l=2 and 5 in l=3. As before, the character representations (3.1) and (3.2) account for the l=3 state of particle 3 at  $P=4\pi/10$ , the l=2 states of particle 3 at  $P=6\pi/10, 8\pi/10, \pi$ , the l=3 states of particle 4 at  $P=6\pi/10, 8\pi/10$  and the l=2 states of particle 4 at  $P=\pi, 12\pi/10, 14\pi/10$ .

We see from Figs. 4 and 5 that the states with l=2 and l=3 exactly fill up the states missing in l=1. We also see that the  $L\to\infty$  excitation curves give an excellent fit to the finite size energies for l=1 for all momenta which are allowed by (2.14). Such agreement is not particularly seen for the remaining states in l=2,3.

In Figs. 6 and 7 we study this same selection of states for q > 0 and compare with the massive dispersion relation (2.3). Here we see that if q is sufficiently large then all states, regardless of their sector, lie on the same massive dispersion curve.

#### 4. Conclusions

We may now discuss the findings of [12]-[13] and the relation with [1]. The study in [12]-[13] is restricted to P=0 and the major conclusions are 1) that the string content of the P=0 states qualitatively changes with q and 2) that the masses of particles 1-5 as determined from P=0 agree with the masses of the  $E_8$  field theory of [7]-[8]. These conclusions are in agreement with our data. On the other hand it is perhaps not exactly

fair to say that the single particle states 1 and 3 lie in sector l=2 and particles 2 and 4 lie in sector l=3 because the *only* sector at q=0 which contains single particle states is l=1. When q increases there is a motion of the finite number of states in l=2,3 to smoothly join the macroscopic number of states in l=1. This is the numerical explanation of why the mass computations done in [1] in the sector l=1 are correct even though P=0 is never in the sector l=1. This motion is seen numerically in [12]-[13] where the crossover happens on a scale of  $q \sim L^{-15/8}$ . It is clearly most desirable to prove this analytically.

We note that the absence of genuine single particle states in l = 2, 3 means that these sectors have a somewhat different nature than l = 1 and it is presumably for this reason that a representation of the spectrum in the form (2.12)–(2.16) has not yet been found. Such a representation is needed to derive and interpret the character formulas (2.10).

We also note that the recognition that the excitations all have an extended Brillouin zone scheme renders obsolete the suggestion in [13] that the model contains massive particles at criticality.

Finally we remark that the question must be raised as to what the introduction of non-integrability (no matter how small) will have on the single particle excitations. In particular the pieces of the spectrum in the extended Brillouin zone which lie far up in the spectrum and cross a very large number of levels when  $q \to 0$  should be expected to decay in some fashion when non-integrability is introduced. This should be relevant to the relation of the  $E_8$  integrable model to the ordinary Ising model in a magnetic field at  $T = T_c$  and remains to be explored.

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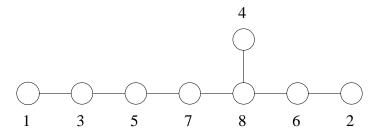


Fig. 1. The Dynkin diagram for the Lie group  $E_8$  with the labeling of nodes used in this paper.

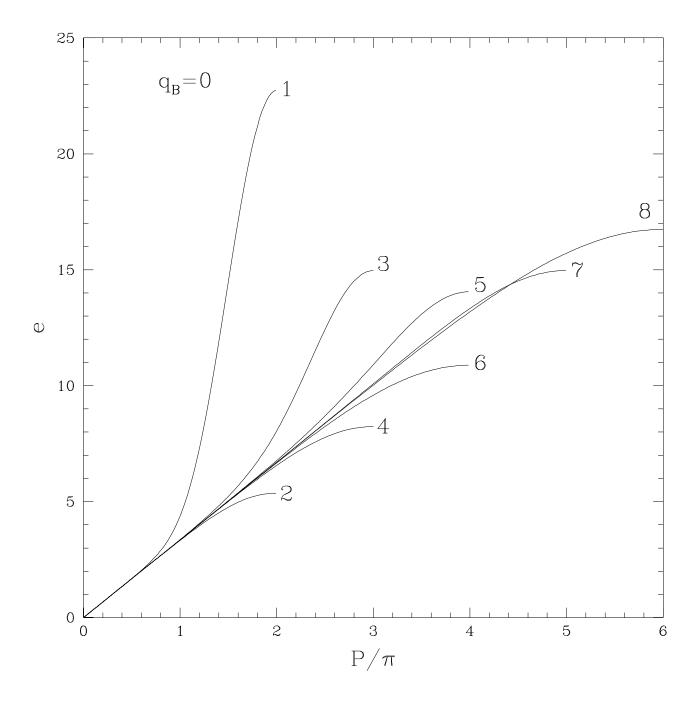


Fig. 2. The eight single particle dispersion relations at  $q_B=0$ .

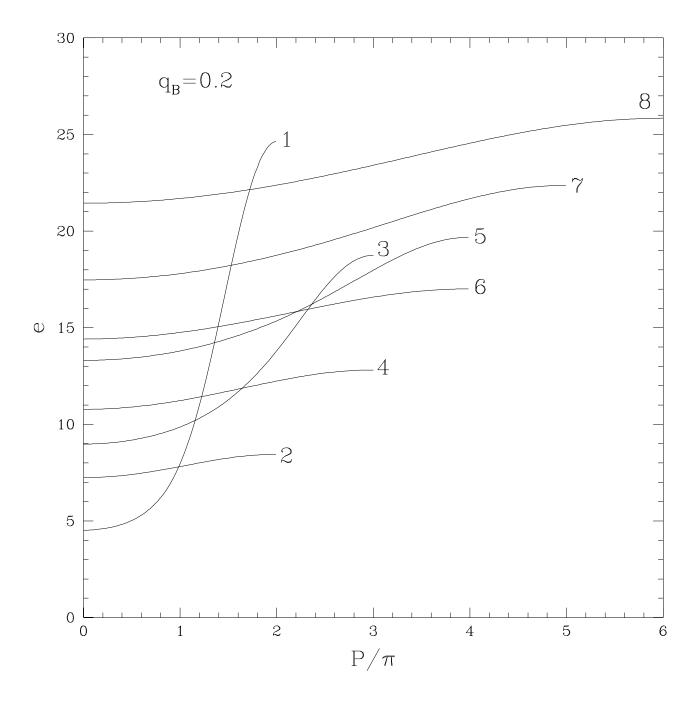


Fig. 3. The eight single particle dispersion relations at  $q_B=0.2$ .

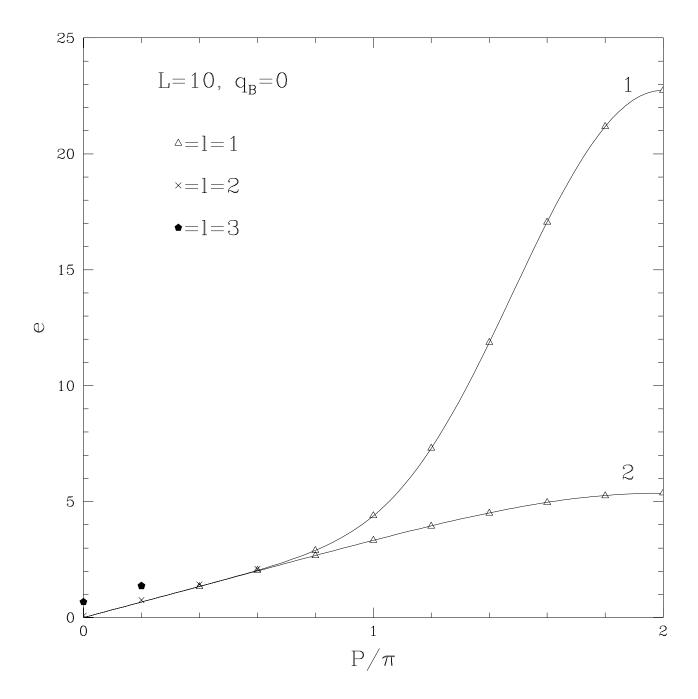


Fig. 4. The plot for  $q_B=0$  and L=10 of the states which for  $q\to 1$  behave as 2A(q). The states in l=1 are marked by triangles, in l=2 by crosses and in l=3 by pentagons. The smooth curves are the theoretical curves (2.1) for particles 1 and 2 for l=1 and  $L\to\infty$ . Note that particle 1 always lies above particle 2.

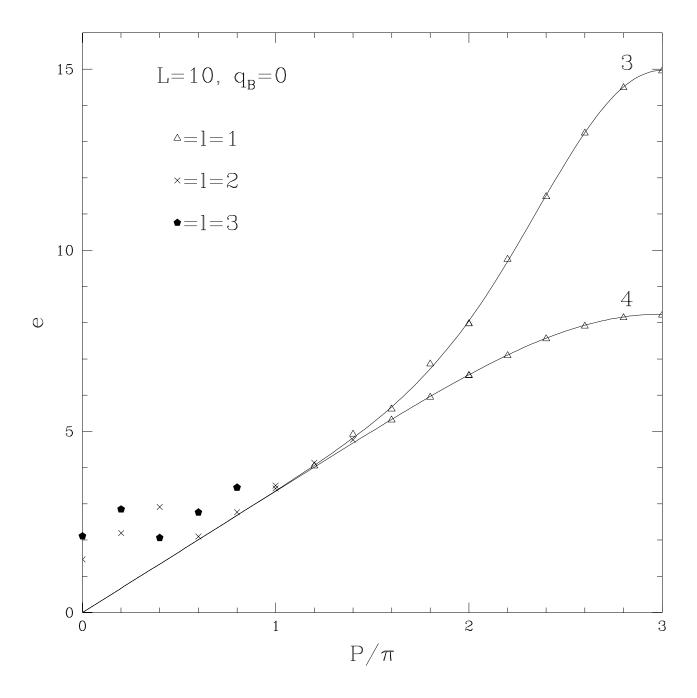


Fig. 5. The plot for  $q_B=0$  and L=10 of the states which for  $q\to 1$  behave as 4A(q). The states in l=1 are marked by triangles, in l=2 by crosses and in l=3 by pentagons. The smooth curves are the theoretical curves (2.1) for particles 3 and 4 for l=1 and  $L\to\infty$ . Note that particle 3 always lies above particle 4.

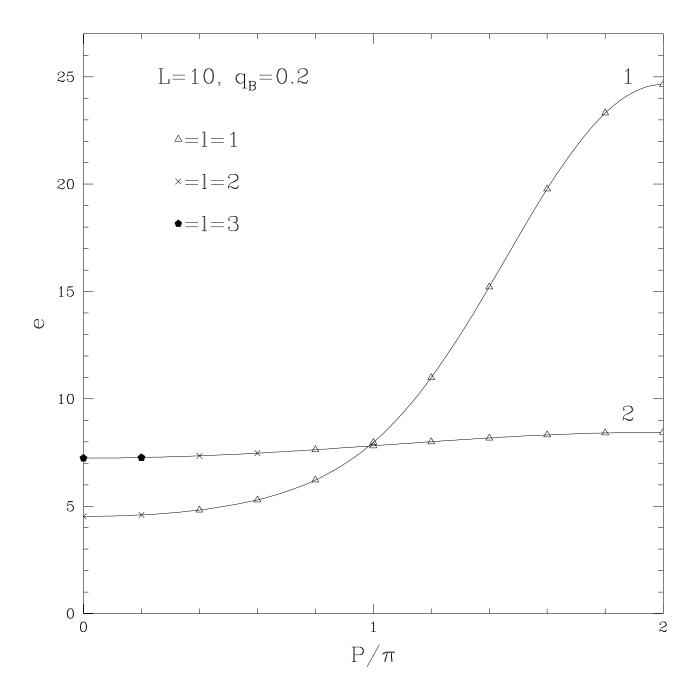


Fig. 6. The plot for  $q_B=0.2$  and L=10 of the states which for  $q\to 1$  behave as 2A(q). The states in l=1 are marked by triangles, in l=2 by crosses and in l=3 by pentagons. The smooth curves are the theoretical curves (2.3) for particles 1 and 2 for l=1 and  $L\to\infty$ . Note that these two curves cross near  $P=\pi$ .

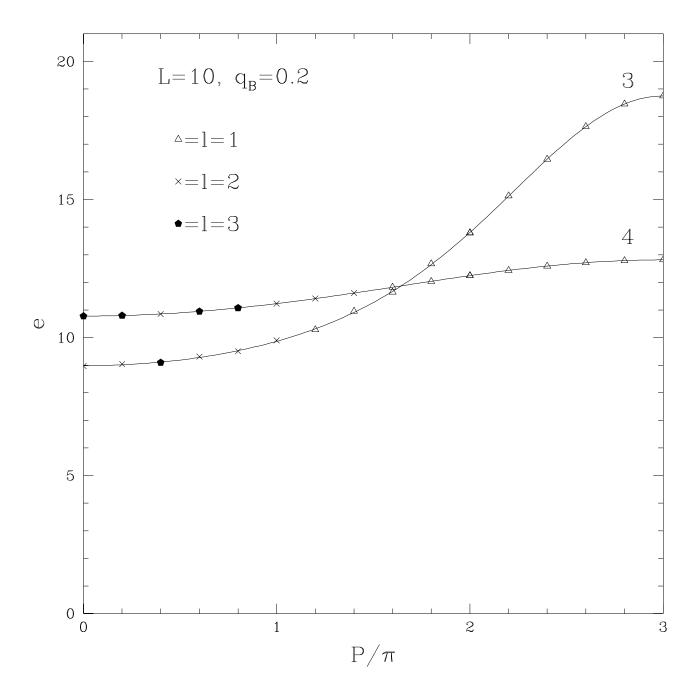


Fig. 7. The plot for  $q_B=0.2$  and L=10 of the states which for  $q\to 1$  behave as 4A(q). The states in l=1 are marked by triangles, in l=2 by crosses and in l=3 by pentagons. The smooth curves are the theoretical curves (2.3) for particles 3 and 4 for l=1 and  $L\to\infty$ . Note that these two curves cross near  $P=1.8\pi$ .

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